One-Dimensional, Steady-State Conduction without Thermal Energy Generation

Chapter Three
Sections 3.1 through 3.4
Methodology of a Conduction Analysis

• Specify appropriate form of the heat equation.
• Solve for the temperature distribution.
• Apply Fourier’s law to determine the heat flux.

Simplest Case: One-Dimensional, Steady-State Conduction with No Thermal Energy Generation.

• Common Geometries:
  – The Plane Wall: Described in rectangular \((x)\) coordinate. Area perpendicular to direction of heat transfer is constant (independent of \(x\)).
  – The Tube Wall: Radial conduction through tube wall.
  – The Spherical Shell: Radial conduction through shell wall.
The Plane Wall

- Consider a plane wall between two fluids of different temperature:

- Implications:
  - Heat flux $q_x$ is independent of $x$.
  - Heat rate $q_x$ is independent of $x$.

- Boundary Conditions: $T(0) = T_{s,1}$, $T(L) = T_{s,2}$

- Temperature Distribution for Constant $k$:
  $$ T(x) = T_{s,1} + \left(T_{s,2} - T_{s,1}\right)\frac{x}{L} $$

- Heat Equation:
  $$ \frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \quad (3.1) $$
Plane Wall (cont.)

- **Heat Flux and Heat Rate:**
  \[ q''_x = -k \frac{dT}{dx} = \frac{k}{L} (T_{s,1} - T_{s,2}) \]  
  \[ q_x = -kA \frac{dT}{dx} = \frac{kA}{L} (T_{s,1} - T_{s,2}) \]  

- **Thermal Resistances** \( R_t = \frac{\Delta T}{q} \) and **Thermal Circuits:**

  Conduction in a plane wall: \( R_{t,\text{cond}} = \frac{L}{kA} \)  

  Convection: \( R_{t,\text{conv}} = \frac{1}{hA} \)  

Thermal circuit for plane wall with adjoining fluids:

\[ R_{tot} = \frac{1}{h_1A} + \frac{L}{kA} + \frac{1}{h_2A} \]  

\[ q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} \]
Plane Wall (cont.)

- **Thermal Resistance for Unit Surface Area:**
  \[ R''_{t,\text{cond}} = \frac{L}{k} \quad R''_{t,\text{conv}} = \frac{1}{h} \]
  
  Units: \( R_t \leftrightarrow \text{K/W} \quad R''_t \leftrightarrow \text{m}^2 \cdot \text{K/W} \)

- **Radiation Resistance:**
  \[ R''_{t,\text{rad}} = \frac{1}{h_r A} \quad R''_{t,\text{rad}} = \frac{1}{h_r} \]
  \[ h_r = \varepsilon \sigma \left( T_s + T_{\text{sur}} \right) \left( T_s^2 + T_{\text{sur}}^2 \right) \]  
  (1.9)

- **Contact Resistance:**
  
  \[ R''_{t,c} = \frac{T_A - T_B}{q_x''} \quad R_{t,c} = \frac{R''_{t,c}}{A_c} \]

Values depend on: Materials A and B, surface finishes, interstitial conditions, and contact pressure (Tables 3.1 and 3.2)
• Composite Wall with Negligible Contact Resistance:

\[ q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t} \]  

(3.14)

\[ \sum R_t = R_{tot} = \frac{1}{A} \left[ \frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_4} \right] = \frac{R''_{tot}}{A} \]

• Overall Heat Transfer Coefficient (U):

A modified form of Newton’s Law of Cooling to encompass multiple resistances to heat transfer.

\[ q_x = U A \Delta T_{overall} \]  

(3.17)

\[ R_{tot} = \frac{1}{UA} \]  

(3.19)
- Series – Parallel Composite Wall:

- Note departure from one-dimensional conditions for $k_F \neq k_G$ .

- Circuits based on assumption of isothermal surfaces normal to $x$ direction or adiabatic surfaces parallel to $x$ direction provide approximations for $q_x$ .
The Tube Wall

- **Heat Equation:**
  \[
  \frac{1}{r} \frac{d}{dr} \left( kr \frac{dT}{dr} \right) = 0
  \]  
  (3.23)

  What does the form of the heat equation tell us about the variation of \( q_r \) with \( r \) in the wall?

  Is the foregoing conclusion consistent with the energy conservation requirement?

  How does \( q''_r \) vary with \( r \)?

- **Temperature Distribution for Constant \( k \):**
  \[
  T(r) = \frac{T_{s,1} - T_{s,2}}{\ln r_2 / r_1} \ln \frac{r}{r_2} + T_{s,2}
  \]  
  (3.26)
• **Heat Flux and Heat Rate:**

\[ q''_r = -k \frac{dT}{dr} = \frac{k}{r \ln \left( \frac{r_2}{r_1} \right)} \left( T_{s,1} - T_{s,2} \right) \]

\[ q'_r = 2\pi r q''_r = \frac{2\pi k}{\ln \left( \frac{r_2}{r_1} \right)} \left( T_{s,1} - T_{s,2} \right) \]

\[ q_r = 2\pi r L q''_r = \frac{2\pi Lk}{\ln \left( \frac{r_2}{r_1} \right)} \left( T_{s,1} - T_{s,2} \right) \tag{3.27} \]

• **Conduction Resistance:**

\[ R_{t,\text{cond}} = \frac{\ln \left( \frac{r_2}{r_1} \right)}{2\pi Lk} \quad \text{Units} \leftrightarrow \text{K/W} \tag{3.28} \]

\[ R'_{t,\text{cond}} = \frac{\ln \left( \frac{r_2}{r_1} \right)}{2\pi k} \quad \text{Units} \leftrightarrow \text{m} \cdot \text{K/W} \]

Why doesn’t a surface area appear in the expressions for the thermal resistance?
Tube Wall (Cont.)

- **Composite Wall with Negligible Contact Resistance**

\[ q_r = \frac{T_{x,1} - T_{x,4}}{R_{tot}} = UA(T_{x,1} - T_{x,4}) \]  \hspace{1cm} (3.30)

Note that \( UA \) is a constant independent of radius.

But, \( U \) itself is tied to specification of an interface.

\[ U_i = \left(A_i R_{tot}\right)^{-1} \]  \hspace{1cm} (3.32)

Note: For the temperature distribution shown, \( k_A > k_B > k_C \).
• **Heat Equation**

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0
\]

What does the form of the heat equation tell us about the variation of \( q_r \) with \( r \)? Is this result consistent with conservation of energy?

How does \( q_r'' \) vary with \( r \) ?

• **Temperature Distribution** for Constant \( k \):

\[
T(r) = T_{s,1} - \left( T_{s,1} - T_{s,2} \right) \frac{1 - \left( r_1/r \right)}{1 - \left( r_1/r_2 \right)}
\]
Spherical Shell (cont.)

- **Heat flux, Heat Rate and Thermal Resistance:**

\[
q''_r = -k \frac{dT}{dr} = \frac{k}{r^2 \left[ \left( \frac{1}{r_1} \right) - \left( \frac{1}{r_2} \right) \right]} (T_{s,1} - T_{s,2})
\]

\[
q_r = 4\pi r^2 q''_r = \frac{4\pi k}{\left( \frac{1}{r_1} \right) - \left( \frac{1}{r_2} \right)} (T_{s,1} - T_{s,2})
\]  \hspace{1cm} (3.35)

\[
R_{t,\text{cond}} = \frac{\left( \frac{1}{r_1} \right) - \left( \frac{1}{r_2} \right)}{4\pi k}
\]  \hspace{1cm} (3.36)

- **Composite Shell:**

\[
q_r = \frac{\Delta T_{\text{overall}}}{R_{tot}} = UA\Delta T_{\text{overall}}
\]

\[
UA = R_{tot}^{-1} \leftrightarrow \text{Constant}
\]

\[
U_i = \left( A_i R_{tot} \right)^{-1} \leftrightarrow \text{Depends on } A_i
\]
Problem 3.23: Assessment of thermal barrier coating (TBC) for protection of turbine blades. Determine maximum blade temperature with and without TBC.

**ASSUMPTIONS:**
1. One-dimensional, steady-state conduction in a composite plane wall,
2. Constant properties,

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**Problem:**

**Thermal Barrier Coating**

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**Schematic:**

- **Zirconia**
  - $R^*_{t,c} = 10^{-4} \text{ m}^2\cdot\text{K/W}$
  - $k = 1.3 \text{ W/m}\cdot\text{K}$

- **Inconel**
  - $k = 25 \text{ W/m}\cdot\text{K}$
  - $T_{max} = 1250 \text{ K}$

- **Boundary Conditions**
  - $h_O = 1000 \text{ W/m}^2\cdot\text{K}$
  - $T_{in,o} = 1700 \text{ K}$
  - $h_J = 500 \text{ W/m}^2\cdot\text{K}$
  - $T_{in,j} = 400 \text{ K}$
**ANALYSIS:** For a unit area, the total thermal resistance with the TBC is

\[
R_{tot,w}^* = h_o^{-1} + (L/k)_Zr + R_{t,c}^* + (L/k)_In + h_i^{-1}
\]

\[
R_{tot,w}^* = \left(10^{-3} + 3.85 \times 10^{-4} + 10^{-4} + 2 \times 10^{-4} + 2 \times 10^{-3}\right) m^2 \cdot K/W = 3.69 \times 10^{-3} m^2 \cdot K/W
\]

With a heat flux of

\[
q_w^* = \frac{T_{\infty,o} - T_{\infty,i}}{R_{tot,w}^*} = \frac{1300 K}{3.69 \times 10^{-3} m^2 \cdot K/W} = 3.52 \times 10^5 W/m^2
\]

the inner and outer surface temperatures of the Inconel are

\[
T_{s,i(w)} = T_{\infty,i} + \left(\frac{q_w^*}{h_i}\right) = 400 K + \left(\frac{3.52 \times 10^5 W/m^2}{500 W/m^2 \cdot K}\right) = 1104 K
\]

\[
T_{s,o(w)} = T_{\infty,i} + \left[\left(\frac{1}{h_i}\right) + (L/k)_In\right] q_w^* = 400 K + \left(2 \times 10^{-3} + 2 \times 10^{-4}\right) m^2 \cdot K/W \left(3.52 \times 10^5 W/m^2\right) = 1174 K
\]
Without the TBC,

\[ R''_{\text{tot,wo}} = h_o^{-1} + \left( \frac{L}{k} \right)_i + h_i^{-1} = 3.20 \times 10^{-3} \text{ m}^2 \cdot \text{K/W} \]

\[ q''_{\omega o} = \left( T_{\infty,o} - T_{\infty,i} \right) / R''_{\text{tot,wo}} = 4.06 \times 10^5 \text{ W/m}^2. \]

the inner and outer surface temperatures of the Inconel are

\[ T_{s,i(\omega o)} = T_{\infty,i} + \left( q''_{\omega o} / h_i \right) = 1212 \text{ K} \]

\[ T_{s,o(\omega o)} = T_{\infty,i} + \left[ \left( 1 / h_i \right) + \left( \frac{L}{k} \right)_i \right] q''_{\omega o} = 1293 \text{ K} \]

Use of the TBC facilitates operation of the Inconel below \( T_{\text{max}} = 1250 \text{ K} \).

**COMMENTS:** Since the durability of the TBC decreases with increasing temperature, which increases with increasing thickness, limits to its thickness are associated with reliability considerations.
Problem 3.62: Suitability of a composite spherical shell for storing radioactive wastes in oceanic waters.

SCHEMATIC:

ASSUMPTIONS: (1) One-dimensional conduction, (2) Steady-state conditions, (3) Constant properties at 300K, (4) Negligible contact resistance.

PROPERTIES: Table A-1, Lead: $k = 35.3 \text{ W/m-K}$, MP = 601K; St.St.: $k = 15.1 \text{ W/m-K}$

ANALYSIS: From the thermal circuit, it follows that

$$q = \frac{T_1 - T_\infty}{\frac{1}{4\pi k r_1} + \frac{1}{4\pi k r_2} + \frac{1}{4\pi k r_3} + \frac{h}{4\pi r_3^2}} = \dot{q} \left[ \frac{4}{3} \pi r_1^3 \right]$$
The thermal resistances are:

\[ R_{\text{Pb}} = \left[ \frac{1}{4\pi \times 35.3 \text{ W/m} \cdot \text{K}} \right] \left[ \frac{1}{0.25\text{ m}} - \frac{1}{0.30\text{ m}} \right] = 0.00150 \text{ K/W} \]

\[ R_{\text{St.St.}} = \left[ \frac{1}{4\pi \times 15.1 \text{ W/m} \cdot \text{K}} \right] \left[ \frac{1}{0.30\text{ m}} - \frac{1}{0.31\text{ m}} \right] = 0.000567 \text{ K/W} \]

\[ R_{\text{conv}} = \left[ \frac{1}{4\pi \times 0.31^2 \text{ m}^2 \times 500 \text{ W/m}^2 \cdot \text{K}} \right] = 0.00166 \text{ K/W} \]

\[ R_{\text{tot}} = 0.00372 \text{ K/W}. \]

The heat rate is then

\[ q = 5 \times 10^5 \text{ W/m}^3 \left( 4\pi / 3 \right) (0.25\text{ m})^3 = 32,725 \text{ W} \]

and the inner surface temperature is

\[ T_1 = T_\infty + R_{\text{tot}}q = 283\text{K} + 0.00372\text{K/W} \left( 32,725 \text{ W} \right) = 405 \text{ K} < \text{MP} = 601\text{K}. \]

Hence, from the thermal standpoint, the proposal is adequate.

**COMMENTS:** In fabrication, attention should be given to maintaining a good thermal contact. A protective outer coating should be applied to prevent long term corrosion of the stainless steel.