Algebra Qualifying Exam, July 2006

Attempt at least **two** *questions from each section. Maximum points can be obtained by answering* **five** *questions correctly, but you may attempt as many questions as you wish. More credit will be given for complete answers than for a number of fragments. All rings are assumed to contain a multiplicative identity.* $\mathbb{N} = \{1, 2, \dots\}$ *and* $\mathbb{Z}$ *denotes the set of integers. For a ring or ideal* R *we let* $M_n(R)$ *denote the set of* $n \times n$ *matrices with coefficients in* R*. All modules will be right modules unless otherwise stated. This exam lasts 4 hours. Good luck!*

Section A

1) (a) Complete the following definition: A ring R is *Noetherian* if ....

(b) Prove that $\mathbb{Q}[x]$, the polynomial ring with coefficients in $\mathbb{Q}$, is Noethereian.

(c) Let I denote the ideal of $\mathbb{Q}[x]$ generated by $f = 6x^2 + 7x + 2$ and $g = 10x^2 + 3x - 1$. Prove that I is not a prime ideal.

(d) Let J denote the ideal of $\mathbb{Q}[x]$ generated by $h = 21x^3 - 3x^2 + 2x + 9$. Show that J is a maximal ideal. (Hint: consider $\mathbb{Z}_2[x]$).

2) (a) Let A be a simple R-module. Prove Schur's lemma, which asserts that $Hom_R(A, A)$ is a division ring.

(b) Let F be a field. Show that $M_2(F)$ is a simple ring.

(c) Let $G \neq 1$ be a finite group. Exhibit a non-trivial idempotent in the group ring $\mathbb{Q}G$.

3) (a) Complete the following definition: A ring R is *Artinian* if ....

(b) Describe the general structure of an Artinian ring R in terms of its nil radical $N = Nil(R)$ and the quotient ring R/N.

(c) Show that for an Artinian ring R, as in (b), we have that
$gl.dim(R) = sup\{pd_R V | \text{where } V \text{is an } R-\text{module such that } VN = 0\}$.

4) (a) Complete the following definition: an R-module P is *projective* if....

(b) Let A and A' be R-modules. We write $A \sim A'$ if and only if there exist projective R-modules P and P' such that $A \oplus P \cong A' \oplus P'$. Show that $\sim$ is an equivalence relation.

(c) Define the projective dimension $pd_R(A)$ of an R-module A and the global dimension $gl.dim(R)$ of R.

(d) Prove that if R is not a Wedderburn ring then $gl.dim(R) = 1 + sup\{pd_R(I)\}$ where I runs through the right ideals of R.

5) Prove that an Artinian ring with no nonzero nilpotent elements and no noncentral idempotents is a division ring.

6) Write a short essay about Wedderburn rings; you might like to include definitions and address general structure theory, simple modules and examples.

Section B

7) (a) Explain what is meant when we say that the group F is a *free group on a set* X.

(b) Explain how to construct a free group on a set X. Proofs are not required but you should give detailed explanations.

(c) Explain why, in a free group F on a free generating set X, every non-identity element has infinite order.

(d) Let $H \leq F$. Use part (c) to prove that if $|F : H|$ is finite then whenever $1 \neq K \leq F$ it is the case that $K \cap H \neq 1$.

8) (a) State the 3 Sylow theorems.

(b) Let G be a group of order p^2q^2, where p, q are primes, $p < q$ and $p \nmid (q^2 - 1)$. Prove that $G = P \times Q$, where $|P| = p^2$ and $|Q| = q^2$. Explain why G must be abelian in this case.

(c) Give (and explain) an example to show that the result of part (b) does not hold when $p \mid (q^2 - 1)$.

9) A group G is called *periodic* if every element of G has finite order and G is called *torsion-free* if every non-trivial element of G has infinite order.

(a) Let $N \lhd G$. Prove that G is periodic if and only if N and G/N are both periodic.

(b) Let I be an index set and, for each $i \in I$, let $N_i \lhd G$ be a periodic normal subgroup of G. Prove that the product $\prod_{i \in I} N_i$ is also periodic.

(c) Prove that in an abelian group the set of elements of finite order in G forms a characteristic subgroup of G.

(d) Show, by constructing examples, that (a) and (b) are false if we replace the word "periodic" by "torsion-free".

10) (a) Give the definitions of the terms (i) nilpotent group (ii) soluble (solvable) group.

(b) Prove that if G is a group and N is a normal subgroup of G then G is soluble if and only if both N and G/N are soluble. Show also that if M, N are soluble normal subgroups of G then NM is also soluble.

(c) Give an example to show that if N is a nilpotent normal subgroup of a group G such that G/N is nilpotent then G need not be nilpotent.

11) (a) Let $\theta : H \longrightarrow \operatorname{Aut} N$ be a homomorphism. Explain what is meant by the semidirect product $G = N \rtimes_\theta H$.

(b) Let $D = \{a/b \in \mathbb{Q} | a \in \mathbb{Z}, b = 2^n \text{ for some } n \in \{0\} \cup \mathbb{N}\}$, an additive subgroup of the rational numbers $\mathbb{Q}$. Prove that there is an automorphism α of D given by $\alpha(d) = d/2$ for all $d \in D$ and that $\theta : C_\infty \longrightarrow \operatorname{Aut} D$ given by $\theta(x) = \alpha$ is a homomorphism. Here C_∞ is the infinite cyclic group generated by x.

(c) Let $G = D \rtimes_\theta \langle \alpha \rangle$. Show that G can be generated by two elements.

(d) Give a proof or a counterexample to the statement: Subgroups of finitely generated groups are finitely generated.