Algebra Qualifying Exam, July 8, 1992

Attempt at least **two** *questions from each section. Maximum points can be obtained by answering* **five** *questions correctly, but you may attempt as many questions as you wish. More credit will be given for complete answers than for a number of fragments. All rings are assumed to contain a multiplicative identity.* $\mathbb{N} = \{1, 2, \dots\}$ *and* $\mathbb{Z}$ *denotes the set of integers.*

Section A

1) a) State the three Sylow Theorems.

b) Suppose that G is a group and N is a normal subgroup of G. Let

$$C_G(N) = \{g \in G \quad | \quad gx = xg \quad \forall \quad x \in G\}.$$

Prove that $G/C_G(N)$ is isomorphic to a subgroup of Aut N, the automorphism group of N.

c) Show that if G is a cyclic group of prime order q then Aut G has exactly $q-1$ elements. Use the results of a) and b) together with this fact to prove: If $p < q$ are distinct primes then there exists a non-abelian group of order pq if and only if $p|(q-1)$.

2) (a) Let G be a group. Explain what is meant when we say "G has the maximal condition on subgroups". (ie. G has max).

(b) Prove:

(1) G has max if and only if every subgroup of G is finitely generated.
(2) If $N \triangleleft G$ then G has max if and only if both N and G/N have max.

(c) Give an example of a finitely generated group which does not have max.

3) (a) Explain what is meant by a free group on a set X. Explain, briefly, how to construct a free group on a set X. Detailed proofs are not required.

(b) Let $G = \langle x, y | x^n = y^2, y^{-1}xy = x^{-1} \rangle$. Find the center, $Z(G)$, and derived subgroup, G'. Determine the isomorphism class of $G/Z(G)$ and write a presentation for $G/Z(G)$ which is derived from the presentation for G.

4) (a) Suppose that G is a divisible abelian group. Show that if G has all its non-trivial elements of infinite order then G is isomorphic to a direct sum of copies of the additive group of rationals. Determine the structure of the additive group of real numbers.

(b) Prove that if H is an infinite abelian group all of whose proper subgroups are finite then $H \cong C_{p^\infty}$.

5) (a) Explain what is meant by the statement: " G is a nilpotent group of class c".

(b) Suppose that G is a finite group. Prove that G is nilpotent if and only if G is the direct product of its Sylow p-subgroups.

(c) Construct an example to show that an infinite group G can be a direct product of its maximal p-subgroups, but that G need not be nilpotent.

Section B

6) (a) Let D be a division ring (ie. a not necessarily commutative field) and let $M_n(D)$ be the ring of all $n \times n$ matrices with entries in D. Prove that $M_n(D)$ is simple.

(b) Sketch a proof of the fact that $M_n(D)$ is left Artinian. You may assume any facts you need from the Jordan-Hölder-Schrier theory of composition series.

7) (a) Complete the following definition: A ring R is primitive if...

(b) Sketch the proof of the following Wedderburn-Artin Theorem: If R is primitive and satisfies DCC on left ideals then $R \cong M_n(D)$ for a suitable division ring D and some $n \in \mathbb{N}$.

(c) Prove that every simple left Artinian ring has the form $M_n(D)$.

8) (a) Prove that $\mathbb{Z}/m\mathbb{Z} \bigotimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$ where (m,n) denotes the greatest common divisor of m and n.

(b) Prove that $A \bigotimes B = 0$ whenever A is a divisible abelian group and B is a periodic abelian group.

9) Recall that a ring R has invariant basis number (IBN) if $R^{(m)} \cong R^{(n)}$ implies $m = n$. It is known that $R^{(m)} \cong R^{(n)}$ if and only if there is an $m \times n$ matrix A and an $n \times m$ matrix B, each with entries in R, such that $AB = I_m$ and $BA = I_n$.

(a) Let $\psi : T \longrightarrow R$ be a ring homomorphism where R has IBN. Prove that T has IBN. Deduce that every commutative ring has IBN. Prove that $M_k(R)$ has IBN for every $k \in \mathbb{N}$.

(b) Let V be the infinite dimensional vector space over a field F with basis $\{v_1, v_2, \dots\}$ and let $S = \mathrm{End}_F(V) = \mathrm{Hom}_F(V,V)$,the endomorphism ring of V. Define $\Phi : S \longrightarrow S^{(2)}$ by defining $\Phi(f) = (f_1, f_2)$ for each $f \in S$ where f_1 and f_2 are given by

$$f_1(v_i) = f(v_{2i}) \quad \text{and} \quad f_2(v_i) = f(v_{2i-1}), \text{for all } i \in \mathbb{N}.$$

Prove that Φ is an isomorphism and deduce that $S^{(m)} \cong S^{(n)}$ for all $m, n \in \mathbb{N}$.

10) Recall that the Jacobson Radical of a ring R is given by $\text{Jac}(R) = \cap$ {primitive ideals of R} $= \cap${maximal left ideals of R}.

(a) Prove that $\text{Jac}(R)$ is the set of all $r \in R$ such that $1 - rs$ is left invertible for all $s \in R$.

(b) Prove that if $a \in \text{Jac}(R)$ then $1 - a$ is a unit.

(c) We call a left ideal A of R <u>small</u> if, for every left ideal B,

$$A + B = R \quad \text{implies} \quad B = R.$$

Show that A is small if and only if $A \subseteq \text{Jac}(R)$.