Algebra Qualifying Exam, July 8, 1996

Maximum points can be obtained by answering **five** *questions correctly. You may attempt* **ten** *questions. More credit will be given for complete answers than for a number of fragments.*

Section A: Rings

1) (i) Construct a field with exactly 125 elements.

(ii) Construct a noncommutative ring that contains exactly 2500 elements.

(iii) Show that the ideal of $\mathbb{Q}[x]$ generated by $f(x) = x^3 - 3x + 2$ and $g(x) = x^4 + 2x^3 - 4x^2 - 2x + 3$ is not prime.

2) (i) State the main results in the fundamental theorem of Galois theory.

(ii) Find the Galois group of $f(x) \in \mathbb{Q}[x]$ where $f(x) = x^3 - 3$.

(iii) Prove that if a and b are algebraic over $\mathbb{Q}$ then so is $a + b$.

3) (i) Complete the following definition: An R-module P is *projective* if ....

Let A and A' be R-modules. We write $A \sim A'$ if and only if there exist projective R-modules P and P' such that $A \bigoplus P \cong A' \bigoplus P'$. It is known that $\sim$ is an equivalence relation.

(ii) Let A be an R-module. Define the projective dimension $\text{pd}_R A$ of A and the global dimension gl dim R of R.

(iii) Prove that gl dim $R = 0$ if and only if R is a Wedderburn ring and that gl dim $R = 0$ if and only if R is hereditary but not Wedderburn.

4) Let I be a nil ideal of the ring R. Prove that,

(i) If $x \in R$ is nilpotent then $1 - x$ is invertible.

(ii) If J/I is a nil ideal of R/I then J is a nil ideal of R.

(iii) An arbitrary sum of nil ideals is nil.

Define the nil radical of R. Describe the structure of an Artinian ring R.

5) (i) Complete the following definition: A ring R is *primitive* if ....

(ii) Sketch a proof of the following theorem: If R is primitive and satisfies the DCC on left ideals (i.e. R is Artinian), then $R \cong M_n(D)$ for a suitable division ring D and some $n \geq 1$.

(iii) Let R=$\text{Tr}(3, \mathbb{Q})$, the ring of all upper-triangular matrices with rational entries. Find the Jacobson radical $\text{Rad}(R)$ of R.

6) (i) Complete the following definition: A ring R is *local* if ....

(ii) Prove that a ring R is local if and only if every element of R that is not contained in $\mathrm{Rad}(R)$ is a unit.

(iii) Let $R = \{\frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0, 5 \nmid b\}$, a subring of $\mathbb{Q}$. Prove that R is local.

7) Let D be a division ring. Prove that $M_2(D)$ is a simple ring that has infinitely many right ideals. Prove that $M_2(R)$ is not simple if R is not a division ring. Prove that $M_2(\mathbb{Z})$ has infinitely many maximal ideals.

8) (i) Let R be a ring and let I be a direct summand of the left R-module ${}_RR$. Prove that $I = Re$ for some $e \in R$ such that $e^2 = e$

Let K be a field of characteristic $p > 0$ and let $a \in K$. Let $f(x) = x^p - x + a$, an element of $K[x]$. Prove that if b is a root of $f(x)$ and c is an element of the prime field of K , then $b + c$ is a root of $f(x)$. Deduce that either $f(x)$ is irreducible over K or K is a splitting field for $f(x)$.

Section B: Groups

9) State the three Sylow theorems. Prove that a group of order 28 has a normal subgroup of order 7. Prove that if G does not have exactly 7 Sylow 2-subgroups then G has a normal subgroup of order 4 and is abelian.

10) Complete the following definition: A group G is *nilpotent of class* c if... .

(i) Let p be a prime. Prove that every finite p-group is nilpotent.

(ii) Prove that, for every $c > 0$, there exists a finite p-group of class c.

(iii) Prove that if G is nilpotent of class c and $N \triangleleft G$, then G/N is nilpotent of class at most c.

11) (i) Prove that a group G satisfies the maximal condition on subgroups if and only if every subgroup of G is finitely generated.

(ii) Expain what is meant by a free group on a set X. Explain how to construct a free group on a set X. Detailed proofs are not required.

12) Let $\theta : H \to AutN$ be a homomorphism. Explain what is meant by the semidirect product $G = N \rtimes_\theta H$. Let p be a prime . Prove that $AutC_p$ has order $p - 1$. Deduce that if q is a prime such that $q|p - 1$, then there exists a nonabelian group of order pq.

13) (i) Prove that every finitely generated subgroup of the additive group of the rational numbers is cyclic. Use this fact to prove that there are no groups G such that $AutG \cong \mathbb{Q}$.

(ii) Let G be a group with a torsion-free normal subgroup N such that G/N is a finite p-group. Let H be a finite subgroup of G. Prove that H is a p-group.