Algebra Qualifying Exam, June 27, 1997

Maximum points can be obtained by answering **five** *questions correctly. You may attempt as many questions as you like. More credit will be given for complete answers than for a number of fragments. All rings have 1. Z(G) denotes the center of the group G and* $\mathbb{N}, \mathbb{Q}$ *stand as usual for the natural numbers and rational numbers respectively*

Section A: Rings

1 (a) Let R be a ring. Complete the following definition. An R-module V is *Noetherian* if ... .
(b) Prove that V is Noetherian if and only if every submodule of V is finitely generated.

2 (a) Give two equivalent definitions of the (Jacobson) radical $\mathrm{Rad}(R)$ of a ring R.
(b) Give an example of a ring R with a subring S such that R is semiprimitive (that is, $\mathrm{Rad}(R) = 0$), but S is not semiprimitive. (Hint: appropriate rings of matrices will work.)
(c) Is a subring of a Wedderburn ring necessarily Wedderburn?

3 (a) Let A be an arbitrary R-module. Give an example of an exact sequence of R-modules

$$0 \to B \to P \to A \to 0$$

where P is projective.
(b) Give an example of a ring R and a projective R-module P that is not a free R-module.

4 (a) What is a prime ring?
(b) What is a primitive ring?
(c) Show every primitive ring is prime, but not conversely.

5 (a) What universal property does the tensor product $A \bigotimes_R B$ of (appropriate) R-modules A and B satisfy?
(b) Give an example of the tensor product of two nontrivial R-modules A and B such that $A \bigotimes_R B = 0$.

6 (a) Explain why every Wedderburn ring is a hereditary ring.
(b) What is a Dedekind domain?
(c) Give an example of Wedderburn ring that is not a Dedekind domain.

7 Provide an example of a ring R with **ONE** of the following properties: (i) R does not have invariant basis number; (ii) R is left Artinian but not right Artinian; or (iii) R is right Noetherian but not left Noetherian. Give a clear but concise justification.

Section B: Groups

8 (a) State the three Sylow theorems.
(b) Let G be a group of order 175. Prove that such a group is never simple and that in fact it is always abelian.

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9 (a) Give the definitions of the terms (i) nilpotent group (ii) soluble (solvable) group.

(b) Prove that if G is a group and N is a normal subgroup of G then G is soluble if and only if both N and G/N are soluble. Show also that if M, N are soluble normal subgroups of G then NM is also soluble.

(c) Give an example to show that if N is a nilpotent normal subgroup of a group G such that G/N is nilpotent then G need not be nilpotent.

10 A group G is called *divisible* if for each $n \in \mathbb{N}$ and each $y \in G$ there exists $x = x(y) \in G$ such that $x^n = y$. Note that we do not assume that G is abelian.

(a) Prove that every non-trivial divisible group is infinite.

(b) Give an example of a non-trivial divisible group, being sure to explain why your example is divisible.

(c) Prove that if G is divisible and $N \triangleleft G$ then G/N is also divisible.

(d) State the structure theorem for finitely generated abelian groups and use this to deduce that if G is a non-trivial finitely generated abelian group then G cannot be divisible.

11 (a) Let $S_1 \leq S_2 \leq S_3 \leq \ldots$ be an infinite increasing chain of simple groups and let $G = \cup_{i=1}^{\infty} S_i$. Prove that G is also a simple group and hence give an example of an infinite simple group.

(b) Prove directly, without using the structure of finitely generated abelian groups, that $\mathbb{Q}$ under addition is not a finitely generated group.

12 (a) Let G be a group and let $\mathrm{Aut}\,(G)$ denote the group of automorphisms of G and as usual let $\mathrm{Inn}\,(G)$ denote the corresponding group of inner automorphisms of G. Prove in detail that $G/Z(G) \cong \mathrm{Inn}\,(G)$.

(b) Prove that if G is a group and $G/Z(G)$ is cyclic then G is abelian and deduce that a non-trivial cyclic group of odd order is never isomorphic to $\mathrm{Aut}\,(G)$ for any group G.

13 Suppose G is a group and $H \leq G$. Let $X = \{Hg : g \in G\}$ be the set of right H-cosets.

(a) Prove that there is a homomorphism from G into $\mathrm{Sym}\,X$, the symmetric group on X.

(b) Prove that if $n \in \mathbb{N}$ and $|G : H| = n$ then G has a normal subgroup K, contained in H such that $|G : K| \leq n!$

(c) Deduce that if G is a finite group and p is the smallest prime dividing the order of G then any subgroup H of index p in G is normal in G.