MATH 126
TAYLOR POLYNOMIALS (Sec.12.12)

1. Taylor polynomials of degree n around a for $f(x)$.

$$T_n(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f(n)(a)}{n!}(x-a)^n.$$

$$f(x)\approx T_n(x)$$

and the error

$$|R_n(x)|=|\ f(x)-P_n(x)|\leq\frac{M}{(n+1)!}|x-a|^{n+1} \text{ where}$$

$$|f^{(n+1)}(x)|\leq M$$

Ex 1. Approximate $f(x)=\sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a=8$.
Give an estimate of $\sqrt[3]{8.5}$
Give an upper bound for the error of the above approximation

Ex 2. Approximate $f(x)=\cos(x)$ by a Taylor polynomial of degree 2 at $a=\frac{\pi}{6}$.
Give an estimate of $\cos(32^o)$
Give an upper bound for the error of the above approximation

2. Taylor polynomials of degree n around 0 for $f(x)$ (MacLaurin polynomials)

$$T_n(x)=f(0)+\frac{f'(0)}{1!}(x-0)+\frac{f''(0)}{2!}(x-0)^2+\ldots+\frac{f(n)(0)}{n!}(x-0)^n \text{ or}$$

$$T_n(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f(n)(0)}{n!}x^n.$$

and the error

$$|R_n(x)|=|\ f(x)-P_n(x)|\leq\frac{M}{(n+1)!}|x|^{n+1} \text{ where}$$

$$|f^{(n+1)}(x)|\leq M$$

Ex 3. (a) Find the MacLaurin polynomial of degree 5 for $f(x)=e^x$.

Ans. $T_5(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}$

(b) Give an estimate of $e^{-0.2}$

Ans. $e^{-0.2}=1-\frac{0.2}{1!}+\frac{(0.2)^2}{2!}-\frac{(0.2)^3}{3!}+\frac{(0.2)^4}{4!}-\frac{(0.2)^5}{5!}$

(c) Give an upper bound for the error of this approximation

Ans. $\frac{(0.2)^6}{6!}$

(d) Give an estimate such that the error is less than 0.001.

Ans. $e^{-0.2} \approx 1 - \frac{0.2}{1!} + \frac{(0.2)^2}{2!} - \frac{(0.2)^3}{3!}$ b/c error $\leq \frac{(0.2)^4}{4!} < 0.001$

Ex 4. (a) Find the MacLaurin polynomial of degree 5 for $f(x) = \sin(x)$

Ans. $T_5(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}$

(b) Give an estimate of $\sin(0.2)$.

Ans. $\sin(0.2) \approx \frac{0.2}{1!} - \frac{(0.2)^3}{3!} + \frac{(0.2)^5}{5!}$

(c) Give an upper bound for the error of this approximation

Ans. $\frac{(0.2)^7}{7!}$

3. What is the Taylor polynomail of degree 1?

Ans. It is the tangent line or the linearization of the function $f(x)$ at a.

TAYLOR SERIES (Sec. 12.10)

4. Taylor series and MacLaurin series.

Imagine that the degree of the Taylor (MacLaurin) polynomial is infinite, the polynomial has infinitely many terms: it becomes a series, called the **Taylor (MacLaurin) series** for the function $f(x)$ at a, and

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{f(n)(a)}{n!}(x-a)^n \quad \textbf{(Taylor series)}$$

or $$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$
$$= \sum_{n=0}^{\infty} \frac{f(n)(0)}{n!}x^n \quad \textbf{(MacLaurin series)}$$

Ex 5. The Maclarin series of $f(x) = e^x$ is

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex 6. The MacLaurin series of $f(x) = \sin(x)$ is:

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{p=0}^{\infty} (-1)^p \frac{x^{2p+1}}{(2p+1)!}$$

Ex 7. The MacLaurin series of $f(x) = \cos(x)$ is:

$$\sin(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{p=0}^{\infty} (-1)^p \frac{x^{2p}}{(2p)!}$$

5. Finding and Using Taylor series.

(a) Use the formulas above to find the MacLaurin series of the following functions:

$f(x) = e^{-x^2}$; $g(x) = sin(x^2)$; $h(x) = cos(x^2)$

Ans. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$ $\quad \sum_{p=0}^{\infty} (-1)^p \frac{x^{4p+2}}{(2p+1)!}$ $\quad \sum_{n=0}^{\infty} (-1)^p \frac{x^{4p}}{(2p)!}$

(b) Find the MacLaurin polynomials of degree 6 of the functions in part (a)

Ans. See part (e) below

(c) Express the following ainderivatives as **power series**:

$\int e^{-x^2}\, dx$ $\quad \int sin(x^2)\, dx$ $\quad \int cos(x^2)\, dx$

Ans. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)} + C$; $\sum_{p=0}^{\infty} (-1)^p \frac{x^{4p+3}}{(2p+1)!(4p+3)} + C$;

$\sum_{n=0}^{\infty} (-1)^p \frac{x^{4p+1}}{(2p)!(4p+1)} + C$

(d) Express the values of the integrals as **series**

$\int_0^{0.5} e^{-x^2}\, dx$ $\quad \int_0^{0.5} sin(x^2)\, dx$ $\quad \int_0^{0.5} cos(x^2)\, dx$

Ans. $\sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2})^{2n+1}}{n!(2n+1)}$; $\sum_{p=0}^{\infty} (-1)^p \frac{(\frac{1}{2})^{4p+3}}{(2p+1)!(4p+3)}$; $\sum_{n=0}^{\infty} (-1)^p \frac{(\frac{1}{2})^{4p+1}}{(2p)!(4p+1)}$

(e) Write the Taylor polynomials of degree 6, $T_6(x)$, for

$f(x) = e^{-x^2}$; $g(x) = sin(x^2)$; $h(x) = cos(x^2)$

Ans. For $f(x) = e^{-x^2}$, $T_6(x) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!}$

For $g(x) = sin(x^2)$, $T_6(x) = x^2 - \frac{x^6}{3!}$

For $h(x) = cos(x^2)$, $T_6(x) = 1 - \frac{x^4}{2!}$

(f) Use above $T_6(x)$ to approximate:

$\int_0^{0.5} e^{-x^2}\, dx$ $\quad \int_0^{0.5} sin(x^2)\, dx$ $\quad \int_0^{0.5} cos(x^2)\, dx$

Ans. $\frac{1}{2} - \frac{1}{3(2^3)1!} + \frac{1}{5(2^5)2!} - \frac{1}{7(2^7)3!}$

$\frac{1}{3(2^3)} - \frac{1}{7(2^7)3!}$

$\frac{1}{2} - \frac{1}{5(2^5)2!}$

(g) Give an upper bound for the error in each approximation above

(Note: These are alternating series).

Ans. $\frac{1}{9(2^9)4!}$; $\frac{1}{11(2^{11})5!}$; $\frac{1}{9(2^9)4!}$

(h) Give an estimate of each definite integral such the error is less than 0.001.

Ans. (1) $\frac{1}{2} - \frac{1}{3(2^3)1!} + \frac{1}{5(2^5)2!}$ b/c $\frac{1}{7(2^7)3!} = \frac{1}{5376} < 0.001$

(2) $\frac{1}{3(2^3)}$ b/c $\frac{1}{7(2^7)3!} = \frac{1}{5376} < 0.001$

(3) $\frac{1}{2} - \frac{1}{5(2^5)2!}$ b/c $\frac{1}{9(2^9)4!} = \frac{1}{110{,}592} < 0.001$

6. Some more MacLaurin series. (SEC. 12.9)

$$\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+x^4+\ldots = \frac{1}{1-x} \quad \text{(for } |x|<1)$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = tan^{-1}(x) \quad \text{(for } |x|<1)$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = ln(1+x) \quad \text{(for } |x|<1)$$